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COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review**. *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on "Abelian sandpile model"

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We reconsider the method for computing a correlation function of any order in the Abelian sandpile model, suggested by Chau in his Rapid Communication [Phys. Rev. E **47**, R3815 (1993)]. Using a counterexample, we show discrepancies in some of the expressions obtained by him.

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In a recent Rapid Communication [1], Chau has considered the Abelian sandpile model with the properties indicating that when a sand grain is added to a lattice site i its height h_i increases to h_i+1 so long as h_i is less than some threshold value; if it is more than the threshold, the site topples and the height h_j changes to $h_j-\Delta_{ij}$ for all j. He writes the following relations [Eqs. (2) and (6) of Ref. (1)]:

$$G_{ij/\bar{k}}=\frac{(\mathrm{adj}\Delta_{(k)})_{ij}}{\det\Delta} .$$

Here $G_{ij/\bar{k}}$ has been defined as the average number of topplings at site j given that a particle is added to site i and that site k does not topple, with $k\neq i$ and $k\neq j$. $\Delta_{(k)}$ is a matrix obtained by deleting column k and row k of Δ. His second relation is

$$P_i(j)=\frac{(\mathrm{adj}\Delta'')_{ij}}{\det\Delta} .$$

Here $P_i(j)$ is the probability of toppling of site j when a particle is added to site i and matrix Δ'' is obtained from Δ using the following procedure. First, the matrix $\tilde{\Delta}$ is generated with $\tilde{\Delta}_{pq}=0$ if $p=j$ and $q\neq j$, $\tilde{\Delta}_{pq}=\Delta_{pq}$ otherwise. This is then partially diagonalized to another matrix Δ' such that the only nonzero entries are in the diagonal elements and in column j. Then matrix Δ'' is generated by replacing element Δ'_{ij} of Δ' by element $\Delta''_{ij}=\max(\Delta'_{ij}, -\Delta'_{jj})$. The purpose of this Comment is to show that both these expressions are in error, and we show this by a counterexample.

Consider a one-dimensional lattice of six sites with open boundary conditions. The toppling matrix Δ is given by

$$\Delta=\begin{vmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & -1 & 2 \end{vmatrix} .$$

There are only seven permitted configurations [2], namely 111111, 111110, 111101, 111011, 110111, 101111, and 011111. We note that Eq. (3a) of Ref. [1] is not correct if $G_{ij/\bar{k}}$ is the conditional expectation as claimed by the author. It is true in general only if $G_{ij/\bar{k}}$ is treated as the expectation value of $\eta_{ij}(1-\chi_{ik})$, where η_{ij} is a random variable which gives the number of topplings at j when a particle is added to i, χ_{ik} is another random variable which is 1 if k topples and 0 otherwise when a particle is added to i. It is now easy to see that if we take $i=2$, $j=4$, and $k=5$, exact enumeration gives $G_{ij/\bar{k}}=\frac{1}{7}$ and the total number of topplings is 1. If we delete the fifth column and row in matrix Δ we get ten recurrent configurations (1111, 1110, 1101, 1011, and 0111, for sites 1 to 4 with either 0 or 1 for the sixth site). It is easy to see that the total number of topplings in this case is 4. The difference exists between the two cases because the deletion of the fifth site creates a few more recurrent configurations in which j topples when a particle is added to i.

For the second case let us take the same lattice with $i=3$ and $j=4$. The matrix Δ'' (which in this case is the same as Δ') is given by

$$\Delta''=\begin{vmatrix} \frac{4}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0\\ 0 & \frac{3}{2} & 0 & -\frac{3}{4} & 0 & 0\\ 0 & 0 & 2 & -\frac{3}{2} & 0 & 0\\ 0 & 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & -\frac{4}{3} & 2 & 0\\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{3}{2} \end{vmatrix} .$$

Again $(\mathrm{adj}\Delta'')_{ij}/\det\Delta=9/7$ comes out to be different from the value $\frac{4}{7}$, which can be obtained by exact enumeration. In general, if we take a one-dimensional lattice of N sites, with open boundary conditions, and let $i=1$, $j=2$, and $k=3$, exact enumeration shows that

$$G_{ij/\bar{k}}=\tfrac{1}{3} ,$$

$$P_i(j)=(N-1)/(N+1) ,$$

while Eqs. (2) and (6) of Ref. [1] give them as $(N-2)/(N+1)$ and $\frac{1}{3}$, respectively.

We believe the problem arises because there is no exact one-to-one correspondence between the number of topplings in the two systems, since the altered system contains many recurrent configurations not present in the original system.

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[1] H. F. Chau, Phys. Rev. E **47**, R3815 (1993).

[2] P. Ruelle and S. Sen, J. Phys. A **25**, L1257 (1992).